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Pure risk premiums under deductibles. A quantitative management in actuarial practice

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Abstract

It is common practice in most insurance lines for the coverage to be restricted by a deductible. In the paper we investigate the influence of deductibles on pure risk premiums. We derive simple but practical formulae for premiums under franchise, fix amount, proportional, limited proportional and disappearing deductibles in terms of the limited expected value function. Next, we apply the results to typical loss distributions, i.e. lognormal, Pareto, Burr, Weibull and gamma. Finally, we analyse a loss data of one of the power companies. We fit distributions to the data and show how the choice of the distribution and a deductible influences the premium.

Keywords: Applied statistics; Data analysis; Non-life insurance; Deductible; Loss distribution; Pure risk premium

1. Introduction

The choice of a deductible which will be incorporated in the contract and the right pricing of premium under the deductible is vital for the insurance industry. The pricing is crucial since too low a price level results in a loss, while with too high rates a company can price itself out of the market. It is the actuary's task to find methods of premium calculation that take into account possible deductibles. In the paper we present new results which can assist in solving these real-world management problems.

The idea of a deductible is, firstly, to reduce claim handling costs by excluding coverage for the often numerous small claims and, secondly, to provide some motivation to the insured to prevent claims, through a limited degree of participation in claim costs (see e.g. [4] and [13]). If we want to describe in detail the reasons for introducing deductibles we have to mention the following 'properties' of a deductible:

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(i) **loss prevention** – as the compensation is reduced by a deductible the retention of the insured is positive. This makes out a good case for avoiding the loss.

(ii) **loss reduction** – the fact that the deductible puts the policyholder at risk of obtaining only partial compensation provides an economic incentive to reduce the extent of the damage.

(iii) **avoidance of small claims where administration costs are dominant** – for small losses, the administration costs will often exceed the loss itself, and hence the insurance company would want the policyholder to pay it himself.

(iv) **premium reduction** – premium reduction can be an important aspect for the policyholders, they may prefer to take a higher deductible to get a lower premium.

In Section 2 we derive formulae for pure risk premiums under franchise, fix amount, proportional, limited proportional and disappearing deductibles in terms of the limited expected value function. In Section 3 we discuss pure risk premiums under the deductibles for the well-known loss distributions, i.e. lognormal, Pareto, Burr, Weibull and gamma (for the distributions see e.g. [5], [7], [9] and [10]). In Section 4 we consider a loss data of Pumped Storage Power Plants Co. (see [14]). We find analytic distribution functions which fit the observed data well and illustrate graphically the influence of the parameters of the discussed deductibles on the pure risk premium.

2. General formulae for premiums under deductibles

Let X be the total (random) monetary amount of some economic risk; we will briefly call X a risk. A premium calculation principle is a rule saying what premium should be assigned to a given risk. In the paper we will apply the simplest premium (calculation principle) which is called pure risk premium, namely the mean of X. It is often applied in life and some mass lines of business in non-life insurance (see [12]). As we know from ruin theory, the pure risk premium without any kind of loading is insufficient since, in the long run, ruin is inevitable even in the case of substantial (though finite) initial reserves. Nevertheless, the pure risk premium can be – and still is – of practical use because, for one thing, in practice the planning horizon is always limited, and for another, because there are indirect ways of loading a premium, e.g. by neglecting interest earnings.

More precisely, let X denote a non-negative random variable describing the size of claim (risk, loss), $F(t)$ and $f(t)$ its distribution and probability distribution functions respectively, and $h(x)$ the payment function corresponding to a deductible. Moreover, we assume that the expected value EX exists. The pure risk premium P (as we consider only pure risk premiums we will henceforth use the term 'premium' meaning 'pure risk premium') is then equal to the expectation, i.e.

$$P = Eh(X). \tag{1}$$

In the case of no deductible the payment function is obviously of the form $h(x) = x$. This means that if the loss is equal to x, the insurer pays the whole claim amount and $P = EX$.

In order to derive formulae for premiums under deductibles let us recall the so-called limited expected value function, namely

$$E(X, x) = \int_0^x yf(y)dy + x(1 - F(x)), \quad x > 0. \tag{2}$$

The value of this function at a point x is equal to the expected value of the random variable X truncated at the point x. The function is a very useful tool for testing the goodness of fit of an analytic distribution function to the observed claim size distribution function, see e.g. [4].

In the following subsections we will consider the most important types of deductibles and derive corresponding premium formulae.

2.1. **Franchise deductible.** One of the deductibles that can be incorporated in the contract is a so-called franchise deductible. In this case the insurer pays the whole claim, if the agreed deductible amount is exceeded. More precisely, under the franchise deductible of a, if the loss is less than a amount the insurer pays nothing, but if the loss equals or exceeds a amount claim is paid in full. This means that the payment function can be described by

$$h_{FD(a)}(x) = \begin{cases} 0, & x < a, \\ x, & \text{otherwise,} \end{cases} \tag{3}$$

see Figure 1.

It is worth noticing that the franchise deductible satisfies properties (i), (iii) and (iv), but not property (ii). This deductible can even work against property (ii) – since if a loss occurs, the policyholder would prefer it to be greater than or equal to the deductible.

The pure risk premium under the franchise deductible can be expressed in terms of the premium in the case of no deductible and the corresponding limited expected value function.

Formula 2.1. *The pure risk premium in the case of the franchise deductible of a is given by*

$$P_{FD(a)} = P - E(X, a) + a(1 - F(a)). \tag{4}$$

The proof follows directly from the form of the payment function given by (3). It can be easily noticed that this premium is a decreasing function of a, when $a = 0$ the premium is equal to the one in the case of no deductible and if a tends to infinity the premium tends to zero.

2.2. **Fixed amount deductible.** An agreement between the insured and the insurer incorporating a deductible b means that the insurer pays only the part of the claim which exceeds the amount b; if the size of the claim falls below this amount, then the claim is not covered by the contract and the insured receives no indemnification. The payment function is thus given by

$$h_{FAD(b)}(x) = \max(0, x - b), \tag{5}$$

see Figure 2.

The fixed amount deductible satisfies all properties mentioned in Section 1.

The following formula expresses the premium in the case of fixed amount deductible in terms of the premium under the franchise deductible.

Formula 2.2. *The pure risk premium in the case of the fixed amount deductible of b is given by*

$$P_{FAD(b)} = P - E(X, b) = P_{FD(b)} - b(1 - F(b)). \tag{6}$$

The above formula may be readily evaluated by means of the payment function given by (5). As previously, the premium is a decreasing function of b, for $b = 0$ it gives the premium in the case of no deductible and if b tends to infinity, it tends to zero.

2.3. **Proportional deductible.** In the case of the proportional deductible of c, where $c \in (0, 1)$, each payment is reduced by $c \cdot 100\%$ (the insurer pays $(1 - c) \cdot 100\%$ of the claim). Consequently, the payment function is given by

$$h_{PD(c)}(x) = (1 - c)x, \tag{7}$$

see Figure 3.

The proportional deductible satisfies properties (i), (ii) and (iv), but not property (iii), as it implies some compensation for even very small claims.

The following formula shows a simple relation between the premium under proportional deductible and the premium in the case of no deductible.

Formula 2.3. *The pure risk premium in the case of the proportional deductible of c, where $c \in (0, 1)$, is given by*

$$P_{PD(c)} = (1 - c)P. \tag{8}$$

The thesis follows directly from the properties of the expectation. Clearly, the premium is a decreasing function of c, $P_{PD(0)} = P$ and $P_{PD(1)} = 0$.

2.4. **Limited proportional deductible.** The proportional deductible is usually combined with a minimum amount deductible so the insurer does not need to handle small claims and with maximum amount deductible to limit the retention of the insured. For the proportional deductible of c with minimum amount m_1 and maximum amount m_2 $(m_1 < m_2)$ the payment function is given by

$$h_{LPD(c,m_1,m_2)}(x) = \begin{cases} 0, & x \leq m_1, \\ x - m_1, & m_1 < x \leq m_1/c, \\ (1-c)x, & m_1/c < x \leq m_2/c, \\ x - m_2, & \text{otherwise}, \end{cases} \tag{9}$$

see Figure 4.

The limited proportional deductible satisfies all properties mentioned in Section 1.

The following formula expresses the premium under the limited proportional deductible in terms of the premium in the case of no deductible and the corresponding limited expected value function.

Formula 2.4. *The pure risk premium in the case of the limited proportional deductible of c with minimum amount m_1 and maximum amount m_2 $(m_1 < m_2)$ is given by*

$$P_{LPD(c,m_1,m_2)} = P - E(X, m_1) + c\left[E\left(X, \frac{m_1}{c}\right) - E\left(X, \frac{m_2}{c}\right)\right]. \tag{10}$$

Proof. Let $Y = h_{LPD(c,m_1,m_2)}(X)$. Therefore, the distribution function of Y can be expressed by

$$F_Y(t) = \begin{cases} 0, & t \leq 0, \\ F(t+m_1), & 0 < t \leq \frac{m_1}{c} - m_1, \\ F(\frac{t}{1-c}), & \frac{m_1}{c} - m_1 < t \leq \frac{m_2}{c} - m_2, \\ F(t+m_2), & \text{otherwise}, \end{cases} \tag{11}$$

thus giving

$$P_{LPD(c,m_1,m_2)} = \int_0^{\frac{m_1}{c}-m_1} xf(x+m_1)dx + \int_{\frac{m_1}{c}-m_1}^{\frac{m_2}{c}-m_2} xf\left(\frac{x}{1-c}\right)\frac{1}{1-c}dx + \int_{\frac{m_2}{c}-m_2}^{\infty} xf(x+m_2)dx. \tag{12}$$

This may be rewritten as

$$\begin{aligned} P_{LPD(c,m_1,m_2)} &= \int_{m_1}^{\frac{m_1}{c}} (z-m_1)f(z)dz + \int_{\frac{m_1}{c}}^{\frac{m_2}{c}} (1-c)zf(z)dz + \int_{\frac{m_2}{c}}^{\infty} (z-m_2)f(z)dz \\ &= -m_1(F(\frac{m_1}{c}) - F(m_1)) - m_2(1 - F(\frac{m_2}{c})) + \int_{m_1}^{\infty} zf(z)dz - c\int_{\frac{m_1}{c}}^{\frac{m_2}{c}} zf(z)dz \quad (13) \\ &= -m_1(F(\frac{m_1}{c}) - F(m_1)) - m_2(1 - F(\frac{m_2}{c})) + P_{FD(m_1)} - c(P_{FD(\frac{m_1}{c})} - P_{FD(\frac{m_2}{c})}). \end{aligned}$$

By means of (4) we can rewrite the above relation as

$$P_{LPD(c,m_1,m_2)} = P - E(X, m_1) + c\left[E\left(X, \frac{m_1}{c}\right) - E\left(X, \frac{m_2}{c}\right)\right]. \tag{14}$$

□

Sometimes only one limitation is incorporated in the contract, i.e. $m_1 = 0$ or $m_2 = \infty$. It is easy to check that the limited proportional deductible with $m_1 = 0$ and $m_2 = \infty$ reduces to the proportional deductible.

2.5. **Disappearing deductible.** In the case of the disappearing deductible the payment depends on the loss in the following way: if the loss is less than an amount of d_1, the insurer pays nothing, if the loss exceeds d_2 ($d_2 > d_1$) amount, the insurer pays the loss in full, if the loss is between d_1 and d_2, then the deducible is reduced linearly between d_1 and d_2. The payment function is thus given by

$$h_{DD(d_1,d_2)}(x) = \begin{cases} 0, & x \le d_1, \\ \frac{d_2(x-d_1)}{d_2-d_1}, & d_1 < x \le d_2, \\ x, & \text{otherwise}, \end{cases} \tag{15}$$

see Figure 5.

This kind of deductible satisfies properties (i), (iii) and (iv), but similarly as in the case of the franchise deductible it works against (ii).

The following formula shows the premium under the disappearing deductible in terms of the premium in the case of no deductible and the corresponding limited expected value function.

Formula 2.5. *The pure risk premium in the case of the disappearing deductible of d_1 and d_2 ($d_1 < d_2$) is given by*

$$P_{DD(d_1,d_2)} = P + \frac{d_1}{d_2 - d_1} E(X, d_2) - \frac{d_2}{d_2 - d_1} E(X, d_1). \tag{16}$$

Proof. Let $Y = h_{DD(d_1,d_2)}(X)$. The distribution function of Y is

$$F_Y(t) = \begin{cases} 0, & t \le 0, \\ F(\frac{t(d_2-d_1)}{d_2} + d_1), & 0 < t \le d_2, \\ F(t), & \text{otherwise}. \end{cases} \tag{17}$$

Then we obtain

$$P_{DD(d_1,d_2)} = \int_0^{d_2} x \frac{d_2 - d_1}{d_2} f\left(\frac{x(d_2 - d_1)}{d_2} + d_1\right) dx + \int_{d_2}^{\infty} x f(x) dx. \tag{18}$$

This can be transformed as

$$\begin{aligned} P_{DD(d_1,d_2)} &= \int_{d_1}^{d_2} \frac{(z - d_1)d_2}{d_2 - d_1} f(z) dz + \int_{d_2}^{\infty} x f(x) dx \\ &= \frac{d_2}{d_2 - d_1} \left[\int_{d_1}^{\infty} x f(x) dx - \int_{d_2}^{\infty} x f(x) dx - d_1(F(d_2) - F(d_1)) \right] + \int_{d_2}^{\infty} x f(x) dx \\ &= \frac{d_2}{d_2 - d_1} \left[P_{FD(d_1)} - P_{FD(d_2)} - d_1(F(d_2) - F(d_1)) \right] + P_{FD(d_2)}. \end{aligned} \tag{19}$$

Substituting (4) from Formula 2.1 in the last expression we obtain the thesis. □

If $d_1 = 0$, then the premium does not depend on d_2 and it becomes the premium in the case of no deductible. If d_2 tends to infinity, then the disappearing deductible reduces to the fix amount deductible of d_1.

3. Premiums under deductibles for given loss distributions

In this section we present mathematical formulae for premiums calculated in the case of deductibles for a number of loss distributions often used in non-life actuarial practice. The lognormal, Pareto, Burr, Weibull and gamma distributions are typical candidates when looking for a suitable analytic distribution which fits the observed data well, see e.g. [1], [3], [5], [8], [9] and [10].

3.1. **Lognormal distribution.** Consider a random variable Z which has the normal distribution. Let $X = \exp Z$. The distribution of X is called a lognormal distribution. The distribution function is given by

$$F(t) = \Phi\left(\frac{\ln t - \mu}{\sigma}\right) = \int_0^t \frac{1}{\sqrt{2\pi}\sigma y} e^{-\frac{1}{2}\left(\frac{\ln y - \mu}{\sigma}\right)^2} dy, \qquad t, \sigma > 0, \quad \mu \in R, \tag{20}$$

where $\Phi(\cdot)$ is the standard normal (with mean 0 and variance 1) distribution function. The lognormal distribution is very useful in modeling of claim costs. It has a thick right tail and fits many situations well. For small σ it resembles a normal distribution, although this is not always desirable.

Theorem 3.1. *For the lognormal distribution defined by (20) the following formulae hold:*

(a) *franchise deductible premium*

$$P_{FD(a)} = e^{\mu+\frac{\sigma^2}{2}} \left(1 - \Phi\left(\frac{\ln a - \mu - \sigma^2}{\sigma}\right)\right), \tag{21}$$

(b) *fixed amount deductible premium*

$$P_{FAD(b)} = e^{\mu+\frac{\sigma^2}{2}} \left(1 - \Phi\left(\frac{\ln b - \mu - \sigma^2}{\sigma}\right)\right) - b\left(1 - \Phi\left(\frac{\ln b - \mu}{\sigma}\right)\right), \tag{22}$$

(c) *proportional deductible premium*

$$P_{PD(c)} = (1-c)e^{\mu+\frac{\sigma^2}{2}}, \tag{23}$$

(d) *limited proportional deductible premium*

$$\begin{aligned} P_{LPD(c,m_1,m_2)} &= e^{\mu+\frac{\sigma^2}{2}} \left(1 - \Phi\left(\frac{\ln m_1 - \mu - \sigma^2}{\sigma}\right)\right) \\ &+ m_1 \left(\Phi\left(\frac{\ln m_1 - \mu}{\sigma}\right) - \Phi\left(\frac{\ln(m_1/c) - \mu}{\sigma}\right)\right) \\ &+ c e^{\mu+\frac{\sigma^2}{2}} \left(\Phi\left(\frac{\ln(m_1/c) - \mu - \sigma^2}{\sigma}\right) - \Phi\left(\frac{\ln(m_2/c) - \mu - \sigma^2}{\sigma}\right)\right) \\ &+ m_2 \left(\Phi\left(\frac{\ln(m_2/c) - \mu}{\sigma}\right) - 1\right), \end{aligned} \tag{24}$$

(e) *disappearing deductible premium*

$$\begin{aligned} P_{DD(d_1,d_2)} &= \frac{e^{\mu+\frac{\sigma^2}{2}}}{d_2-d_1}\left(d_2-d_1+d_1\Phi\left(\frac{\ln d_2-\mu-\sigma^2}{\sigma}\right)-d_2\Phi\left(\frac{\ln d_1-\mu-\sigma^2}{\sigma}\right)\right) \\ &+ \frac{d_1 d_2}{d_2-d_1}\left(\Phi\left(\frac{\ln d_1-\mu}{\sigma}\right)-\Phi\left(\frac{\ln d_2-\mu}{\sigma}\right)\right). \end{aligned} \tag{25}$$

Proof. For the lognormal distribution defined by (20) the premium in the case of no deductible is given by

$$P = EX = e^{\mu+\frac{\sigma^2}{2}} \tag{26}$$

and the limited expected value function is of the form

$$E(X,x) = e^{\mu+\frac{\sigma^2}{2}}\Phi\left(\frac{\ln x-\mu-\sigma^2}{\sigma}\right)+x\left(1-\Phi\left(\frac{\ln x-\mu}{\sigma}\right)\right), \tag{27}$$

see [6].

Substituting (20), (26), and (27) for $x = a$, in Formula 2.1 we obtain (a) part by simple calculations:

$$\begin{aligned} P_{FD(a)} &= e^{\mu+\frac{\sigma^2}{2}} - e^{\mu+\frac{\sigma^2}{2}}\Phi\left(\frac{\ln x-\mu-\sigma^2}{\sigma}\right) - a\left(1-\Phi\left(\frac{\ln a-\mu}{\sigma}\right)\right)+a\left(1-\Phi\left(\frac{\ln a-\mu}{\sigma}\right)\right) \\ &= e^{\mu+\frac{\sigma^2}{2}}\left(1-\Phi\left(\frac{\ln a-\mu-\sigma^2}{\sigma}\right)\right), \end{aligned} \tag{28}$$

Next, by Formula 2.2, it is easy to get (b) part and from (26) and Formula 2.3 we obtain (c) part.

In order to get (d) part we substitute (26) and (27) in Formula 2.4, and we find

$$\begin{aligned} P_{LPD(c,m_1,m_2)} &= e^{\mu+\frac{\sigma^2}{2}} - e^{\mu+\frac{\sigma^2}{2}}\Phi\left(\frac{\ln m_1-\mu-\sigma^2}{\sigma}\right) - m_1\left(1-\Phi\left(\frac{\ln m_1-\mu}{\sigma}\right)\right) \\ &+ c[e^{\mu+\frac{\sigma^2}{2}}\Phi\left(\frac{\ln(m_1/c)-\mu-\sigma^2}{\sigma}\right)+\frac{m_1}{c}\left(1-\Phi\left(\frac{\ln(m_1/c)-\mu}{\sigma}\right)\right) \\ &- e^{\mu+\frac{\sigma^2}{2}}\Phi\left(\frac{\ln(m_2/c)-\mu-\sigma^2}{\sigma}\right)-\frac{m_2}{c}\left(1-\Phi\left(\frac{\ln(m_2/c)-\mu-\sigma^2}{\sigma}\right)\right)] \\ &= e^{\mu+\frac{\sigma^2}{2}}\left(1-\Phi\left(\frac{\ln m_1-\mu-\sigma^2}{\sigma}\right)\right) \\ &+ m_1\left(\Phi\left(\frac{\ln m_1-\mu}{\sigma}\right)-\Phi\left(\frac{\ln(m_1/c)-\mu}{\sigma}\right)\right) \\ &+ ce^{\mu+\frac{\sigma^2}{2}}\left(\Phi\left(\frac{\ln(m_1/c)-\mu-\sigma^2}{\sigma}\right)-\Phi\left(\frac{\ln(m_2/c)-\mu-\sigma^2}{\sigma}\right)\right) \\ &+ m_2\left(\Phi\left(\frac{\ln(m_2/c)-\mu}{\sigma}\right)-1\right). \end{aligned} \tag{29}$$

Finally, (e) part (the premium under the decreasing deductible) results from Formula 2.5 and (27):

$$\begin{aligned} P_{DD(d_1,d_2)} &= e^{\mu+\frac{\sigma^2}{2}} + \frac{d_1}{d_2-d_1}\left[e^{\mu+\frac{\sigma^2}{2}}\Phi\left(\frac{\ln d_2-\mu-\sigma^2}{\sigma}\right)+d_2\left(1-\Phi\left(\frac{\ln d_2-\mu}{\sigma}\right)\right)\right] \\ &- \frac{d_2}{d_2-d_1}\left[e^{\mu+\frac{\sigma^2}{2}}\Phi\left(\frac{\ln d_1-\mu-\sigma^2}{\sigma}\right)+d_1\left(1-\Phi\left(\frac{\ln d_1-\mu}{\sigma}\right)\right)\right] \\ &= \frac{e^{\mu+\frac{\sigma^2}{2}}}{d_2-d_1}\left(d_2-d_1+d_1\Phi\left(\frac{\ln d_2-\mu-\sigma^2}{\sigma}\right)-d_2\Phi\left(\frac{\ln d_1-\mu-\sigma^2}{\sigma}\right)\right) \\ &+ \frac{d_1 d_2}{d_2-d_1}\left(\Phi\left(\frac{\ln d_1-\mu}{\sigma}\right)-\Phi\left(\frac{\ln d_2-\mu}{\sigma}\right)\right). \end{aligned} \tag{30}$$

□

3.2. **Pareto distribution.** Rather popular is the Pareto distribution which is defined by the formula

$$F(t) = 1 - \left(\frac{\lambda}{\lambda+t}\right)^{\alpha} \qquad t,\alpha,\lambda > 0. \tag{31}$$

The first parameter α controls how heavy tail the distribution has: the smaller the α, the heavier the tail. The expectation of Pareto distribution exists only for $\alpha > 1$.

Theorem 3.2. *For the Pareto distribution defined by (31) with $\alpha > 1$ the following formulae hold:*

(a) *franchise deductible premium*

$$P_{FD(a)} = \frac{1}{\alpha-1}(a\alpha+\lambda)\left(\frac{\lambda}{a+\lambda}\right)^{\alpha}, \tag{32}$$

(b) *fixed amount deductible premium*

$$P_{FAD(b)} = \frac{1}{\alpha-1}(b+\lambda)\left(\frac{\lambda}{b+\lambda}\right)^{\alpha}, \tag{33}$$

(c) *proportional deductible premium*

$$P_{PD(c)} = (1-c)\frac{\lambda}{\alpha-1}, \tag{34}$$

(d) *limited proportional deductible premium*

$$\begin{aligned} P_{LPD(c,m_1,m_2)} &= \frac{1}{\alpha-1}(m_1+\lambda)\left(\frac{\lambda}{m_1+\lambda}\right)^{\alpha} \\ &+ \frac{c}{\alpha-1}\left(\left(\frac{m_2}{c}+\lambda\right)\left(\frac{\lambda}{\frac{m_2}{c}+\lambda}\right)^{\alpha}-\left(\frac{m_1}{c}+\lambda\right)\left(\frac{\lambda}{\frac{m_1}{c}+\lambda}\right)^{\alpha}\right), \end{aligned} \tag{35}$$

(e) *disappearing deductible premium*

$$P_{Z(d_1,d_2)} = \frac{1}{(\alpha-1)(d_2-d_1)}\left(d_2(d_1+\lambda)\left(\frac{\lambda}{d_1+\lambda}\right)^{\alpha}-d_1(d_2+\lambda)\left(\frac{\lambda}{d_2+\lambda}\right)^{\alpha}\right). \tag{36}$$

Proof. When $\alpha > 1$ the expectation of the Pareto distribution defined by (31) exists and the premium under no deductible is

$$P = EX = \frac{\lambda}{\alpha-1}. \tag{37}$$

The limited expected value function is of the form

$$E(X,x)=\frac{\lambda-(x+\lambda)\left(\frac{\lambda}{\lambda+x}\right)^{\alpha}}{\alpha-1}. \tag{38}$$

By virtue of Formula 2.1, formula (37) and (38) we obtain (a) part:

$$\begin{aligned} P_{FD(a)} &= \frac{\lambda}{\alpha-1}-\frac{\lambda-(c+\lambda)\left(\frac{\lambda}{\lambda+c}\right)^{\alpha}}{\alpha-1}+c\left(1-1+\left(\frac{\lambda}{\lambda+c}\right)^{\alpha}\right) \\ &= \frac{1}{\alpha-1}(a\alpha+\lambda)\left(\frac{\lambda}{a+\lambda}\right)^{\alpha}, \end{aligned} \tag{39}$$

which by Formula 2.2 yields also (b) part. As the premium in the case of no deductible is given by (37), (c) part results immediately from Formula 2.3.

In order to obtain (d) part (the case of the limited proportional deductible) we substitute (38) in Formula 2.4:

$$\begin{aligned} P_{LPD(c,m_1,m_2)} &= \frac{\lambda}{\alpha-1}-\frac{\lambda-(m_1+\lambda)\left(\frac{\lambda}{\lambda+m_1}\right)^{\alpha}}{\alpha-1} \\ &+ c\left[\frac{\lambda-\left(\frac{m_1}{c}+\lambda\right)\left(\frac{\lambda}{\lambda+\frac{m_1}{c}}\right)^{\alpha}}{\alpha-1}-\frac{\lambda-\left(\frac{m_2}{c}+\lambda\right)\left(\frac{\lambda}{\lambda+\frac{m_2}{c}}\right)^{\alpha}}{\alpha-1}\right] \\ &= \frac{1}{\alpha-1}(m_1+\lambda)\left(\frac{\lambda}{m_1+\lambda}\right)^{\alpha} \\ &+ \frac{c}{\alpha-1}\left(\left(\frac{m_2}{c}+\lambda\right)\left(\frac{\lambda}{\frac{m_2}{c}+\lambda}\right)^{\alpha}-\left(\frac{m_1}{c}+\lambda\right)\left(\frac{\lambda}{\frac{m_1}{c}+\lambda}\right)^{\alpha}\right). \end{aligned} \tag{40}$$

In order to calculate the last part (e) we substitute (38) in Formula 2.5:

$$\begin{aligned} P_{Z(d_1,d_2)} &= \frac{\lambda}{\alpha-1}+\frac{d_1}{d_2-d_1}\frac{\lambda-(d_2+\lambda)\left(\frac{\lambda}{\lambda+d_2}\right)^{\alpha}}{\alpha-1}-\frac{d_2}{d_2-d_1}\frac{\lambda-(d_1+\lambda)\left(\frac{\lambda}{\lambda+d_1}\right)^{\alpha}}{\alpha-1} \\ &= \frac{1}{(\alpha-1)(d_2-d_1)}\left((d_1+\lambda)\left(\frac{\lambda}{d_1+\lambda}\right)^{\alpha}-(d_2+\lambda)\left(\frac{\lambda}{d_2+\lambda}\right)^{\alpha}\right). \end{aligned} \tag{41}$$

□

3.3. **Burr distribution.** Experience has shown that the Pareto formula is often an appropriate model for the claim size distribution, particularly where exceptionally large claims may occur, see e.g. [4]. However, there is sometimes a need to find heavy tailed distributions which offer greater flexibility than the Pareto law. Such flexibility is provided by the Burr distribution which distribution function is given by

$$F(t)=1-\left(\frac{\lambda}{\lambda+t^{\tau}}\right)^{\alpha} \qquad t,\alpha,\lambda,\tau>0, \tag{42}$$

which is just a generalization of the Pareto distribution. Its mean exists only for $\alpha\tau>1$.

Theorem 3.3. *For the Burr distribution defined by (42) with $\alpha\tau>1$ the following formulae hold:*

(a) *franchise deductible premium*

$$P_{FD(a)} = \frac{\lambda^{\frac{1}{\tau}}\Gamma(\alpha-\frac{1}{\tau})\Gamma(1+\frac{1}{\tau})}{\Gamma(\alpha)}\left(1-\beta\left(1+\frac{1}{\tau},\alpha-\frac{1}{\tau};\frac{a^\tau}{\lambda+a^\tau}\right)\right), \tag{43}$$

(b) *fixed amount deductible premium*

$$P_{FAD(b)} = \frac{\lambda^{\frac{1}{\tau}}\Gamma(\alpha-\frac{1}{\tau})\Gamma(1+\frac{1}{\tau})}{\Gamma(\alpha)}\left(1-\beta\left(1+\frac{1}{\tau},\alpha-\frac{1}{\tau};\frac{b^\tau}{\lambda+b^\tau}\right)\right)-b\left(\frac{\lambda}{\lambda+b^\tau}\right)^\alpha, \tag{44}$$

(c) *proportional deductible premium*

$$P_{PD(c)} = (1-c)\frac{\lambda^{\frac{1}{\tau}}\Gamma(\alpha-\frac{1}{\tau})\Gamma(1+\frac{1}{\tau})}{\Gamma(\alpha)}, \tag{45}$$

(d) *limited proportional deductible premium*

$$\begin{aligned} P_{LPD(c,m_1,m_2)} &= \frac{\lambda^{\frac{1}{\tau}}\Gamma(\alpha-\frac{1}{\tau})\Gamma(1+\frac{1}{\tau})}{\Gamma(\alpha)}\Big[1-\beta\left(1+\frac{1}{\tau},\alpha-\frac{1}{\tau};\frac{m_1^\tau}{\lambda+m_1^\tau}\right) \\ &+ c\beta\left(1+\frac{1}{\tau},\alpha-\frac{1}{\tau};\frac{(\frac{m_1}{c})^\tau}{\lambda+(\frac{m_1}{c})^\tau}\right)-c\beta\left(1+\frac{1}{\tau},\alpha-\frac{1}{\tau};\frac{(\frac{m_2}{c})^\tau}{\lambda+(\frac{m_2}{c})^\tau}\right)\Big] \\ &- m_1\left(\frac{\lambda}{\lambda+m_1^\tau}\right)^\alpha+m_1\left(\frac{\lambda}{\lambda+(\frac{m_1}{c})^\tau}\right)^\alpha-m_2\left(\frac{\lambda}{\lambda+(\frac{m_2}{c})^\tau}\right)^\alpha, \end{aligned} \tag{46}$$

(e) *disappearing deductible premium*

$$\begin{aligned} P_{DD(d_1,d_2)} &= \frac{\lambda^{\frac{1}{\tau}}\Gamma(\alpha-\frac{1}{\tau})\Gamma(1+\frac{1}{\tau})}{\Gamma(\alpha)} \\ &* \frac{\left(d_2-d_1+d_1\beta\left(1+\frac{1}{\tau},\alpha-\frac{1}{\tau};\frac{d_2^\tau}{\lambda+d_2^\tau}\right)-d_2\beta\left(1+\frac{1}{\tau},\alpha-\frac{1}{\tau};\frac{d_1^\tau}{\lambda+d_1^\tau}\right)\right)}{d_2-d_1} \\ &+ \frac{\lambda^{\frac{1}{\tau}}\Gamma(\alpha-\frac{1}{\tau})\Gamma(1+\frac{1}{\tau})}{\Gamma(\alpha)}\frac{d_1d_2}{d_2-d_1}\left(\left(\frac{\lambda}{\lambda+d_2^\tau}\right)^\alpha-\left(\frac{\lambda}{\lambda+d_1^\tau}\right)^\alpha\right). \end{aligned} \tag{47}$$

Sketch of the proof. When $\alpha\tau > 1$ the mean value of the Burr distribution defined by (42) exists and the premium under no deductible is given by

$$P = EX = \frac{\lambda^{\frac{1}{\tau}}\Gamma\left(\alpha-\frac{1}{\tau}\right)\Gamma\left(1+\frac{1}{\tau}\right)}{\Gamma(\alpha)}. \tag{48}$$

Moreover, the limited expected value function is

$$E(X,x) = \frac{\alpha\lambda^{\frac{1}{\tau}}\Gamma\left(\alpha-\frac{1}{\tau}\right)\Gamma\left(1+\frac{1}{\tau}\right)}{\Gamma(\alpha+1)}\beta\left(1+\frac{1}{\tau},\alpha-\frac{1}{\tau};\frac{x^\tau}{\lambda+x^\tau}\right)+x\left(\frac{\lambda}{\lambda+x^\tau}\right)^\alpha, \tag{49}$$

where

$$\Gamma(a) = \int_0^\infty y^{a-1}e^{-y}dy, \qquad \beta(a,b;x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\int_0^x y^{a-1}(1-y)^{b-1}dy. \tag{50}$$

Now, (a)–(e) parts can be proved by virtue of Formulas 2.1–2.5 in an analogous way as in the proof of Theorem 3.2. □

3.4. **Weibull distribution.** Another frequently used analytic claim size distribution is the Weibull distribution which is defined by

$$F(t) = 1 - e^{-(\frac{t}{\beta})^{\alpha}}, \quad t, \alpha, \beta > 0. \tag{51}$$

Theorem 3.4. *For the Weibull distribution defined by (51) the following formulae hold:*

(a) *franchise deductible premium*

$$P_{FD(a)} = \beta\Gamma\left(1 + \frac{1}{\alpha}\right)\left(1 - \Gamma\left(1 + \frac{1}{\alpha}, \left(\frac{a}{\beta}\right)^{\alpha}\right)\right), \tag{52}$$

(b) *fixed amount deductible premium*

$$P_{FAD(b)} = \beta\Gamma\left(1 + \frac{1}{\alpha}\right)\left(1 - \Gamma\left(1 + \frac{1}{\alpha}, \left(\frac{b}{\beta}\right)^{\alpha}\right)\right) - be^{-(\frac{b}{\beta})^{\alpha}}, \tag{53}$$

(c) *proportional deductible premium*

$$P_{PD(c)} = (1 - c)\beta\Gamma\left(1 + \frac{1}{\alpha}\right), \tag{54}$$

(d) *limited proportional deductible premium*

$$\begin{aligned} P_{LPD(c,m_1,m_2)} &= \beta\Gamma\left(1 + \frac{1}{\alpha}\right)\left(1 - \Gamma\left(1 + \frac{1}{\alpha}, \left(\frac{m_1}{\beta}\right)^{\alpha}\right)\right) \\ &+ \beta\Gamma\left(1 + \frac{1}{\alpha}\right)c\left(\Gamma\left(1 + \frac{1}{\alpha}, \left(\frac{m_1}{c\beta}\right)^{\alpha}\right) - \Gamma\left(1 + \frac{1}{\alpha}, \left(\frac{m_2}{c\beta}\right)^{\alpha}\right)\right) \\ &- m_1 e^{-(\frac{m_1}{\beta})^{\alpha}} + m_1 e^{-(\frac{m_1}{c\beta})^{\alpha}} - m_2 e^{-(\frac{m_2}{c\beta})^{\alpha}}, \end{aligned} \tag{55}$$

(e) *disappearing deductible premium*

$$\begin{aligned} P_{DD(d_1,d_2)} &= \frac{\beta\Gamma\left(1 + \frac{1}{\alpha}\right)}{d_2 - d_1}\left(d_2 - d_1 + d_1\Gamma\left(1 + \frac{1}{\alpha}, \left(\frac{d_2}{\beta}\right)^{\alpha}\right) - d_2\Gamma\left(1 + \frac{1}{\alpha}, \left(\frac{d_1}{\beta}\right)^{\alpha}\right)\right) \\ &+ \frac{d_1 d_2}{d_2 - d_1}\left(e^{-(\frac{d_2}{\beta})^{\alpha}} - e^{-(\frac{d_1}{\beta})^{\alpha}}\right). \end{aligned} \tag{56}$$

Sketch of the proof. For the Weibull distribution defined by (51) the premium in the case of no deductible equals

$$P = EX = \beta\Gamma\left(1 + \frac{1}{\alpha}\right) \tag{57}$$

and the limited expected value function is of the form

$$E(X, x) = \beta\Gamma\left(1 + \frac{1}{\alpha}\right)\Gamma\left(1 + \frac{1}{\alpha}, \left(\frac{x}{\beta}\right)^{\alpha}\right) + xe^{-(\frac{x}{\beta})^{\alpha}}, \tag{58}$$

where

$$\Gamma(a, x) = \frac{1}{\Gamma(a)}\int_0^x y^{a-1}e^{-y}dy. \tag{59}$$

Now, (a)–(e) parts result from Formulas 2.1–2.5 in a similar way as in the proof of Theorem 3.2.

□

3.5. **Gamma distribution.** All four presented above distributions suffer from some mathematical drawbacks, e.g. lack of a closed form representation for the Laplace transform and nonexistence of the moment generating function. The gamma distribution given by

$$F(t) = F(t, \alpha, \beta) = \int_0^t \frac{1}{\Gamma(\alpha)\beta^\alpha} y^{\alpha-1} e^{-\frac{y}{\beta}} dy, \qquad t, \alpha, \beta > 0 \tag{60}$$

does not have these drawbacks. It is one of the most important distributions for modeling (not only insurance claims) because it has very tractable mathematical properties and is related to other distributions, cf. [8].

Theorem 3.5. *For the gamma distribution defined by (60) the following formulae hold:*

(a) *franchise deductible premium*

$$P_{FD(a)} = \alpha\beta(1 - F(a, \alpha+1, \beta)), \tag{61}$$

(b) *fixed amount deductible premium*

$$P_{FAD(b)} = \alpha\beta(1 - F(b, \alpha+1, \beta)) - b(1 - F(b, \alpha, \beta)), \tag{62}$$

(c) *proportional deductible premium*

$$P_{PD(c)} = (1-c)\alpha\beta, \tag{63}$$

(d) *limited proportional deductible premium*

$$\begin{aligned} P_{LPD(c,m_1,m_2)} &= \alpha\beta(1 - F(m_1, \alpha+1, \beta)) \\ &+ c\alpha\beta\left(F\left(\frac{m_1}{c}, \alpha+1, \beta\right) - F\left(\frac{m_2}{c}, \alpha+1, \beta\right)\right) \\ &+ m_1\left(F(m_1, \alpha, \beta) - F\left(\frac{m_1}{c}, \alpha, \beta\right)\right) - m_2\left(1 - F\left(\frac{m_2}{c}, \alpha, \beta\right)\right), \end{aligned} \tag{64}$$

(e) *disappearing deductible premium*

$$\begin{aligned} P_{DD(d_1,d_2)} &= \frac{\alpha\beta}{d_2 - d_1}\left(d_2(1 - F(d_1, \alpha+1, \beta)) - d_1(1 - F(d_2, \alpha+1, \beta))\right) \\ &+ \frac{d_1 d_2}{d_2 - d_1}(F(d_1, \alpha, \beta) - F(d_2, \alpha, \beta)). \end{aligned} \tag{65}$$

Sketch of the proof. For the gamma distribution given by formula (60) the premium under no deductible is

$$P = EX = \alpha\beta. \tag{66}$$

Its corresponding limited expected value function can be expressed in terms of the distribution function:

$$
\begin{aligned}
E(X,x) &= \int_0^x y \frac{1}{\Gamma(\alpha)\beta^{\alpha}} y^{\alpha-1} e^{-\frac{y}{\beta}} dy + x(1 - F(x,\alpha,\beta)) \\
&= \frac{\beta\Gamma(\alpha+1)}{\Gamma(\alpha)} \int_0^x \frac{1}{\Gamma(\alpha+1)\beta^{\alpha+1}} y^{\alpha} e^{-\frac{y}{\beta}} dy + x(1 - F(x,\alpha,\beta)) \\
&= \alpha\beta F(x,\alpha+1,\beta) + x(1 - F(x,\alpha,\beta)). \qquad (67)
\end{aligned}
$$

Substituting formulae (66) and (67) in Formulas 2.1–2.5 one can obtain (a)–(e) parts in an analogous way as in the proof of Theorem 3.2.

□

4. Example

The derivation of claim size distributions from the claim data could be considered to be a separate discipline in its own right, which requires applying methods of mathematical statistics, cf. [4]. The objective is to find a distribution function F which fits the observed data in a satisfactory manner. The approach most frequently adopted in the actuarial literature is to find a suitable analytic expression which fits the observed data well and which is easy to handle, see e.g. [3].

Once the distribution is selected, we must obtain parameter estimates. In what follows we use the moment and maximum likelihood estimation. The next step is to test whether the fit is adequate. This is usually done by comparing the fitted and empirical distribution functions, more precisely, by checking whether values of the fitted distribution function at sample points form a uniform distribution, cf. [2]. To this end we applied the well- and not so well-known non-parametric tests, namely χ^2, Kolmogorov–Smirnov (KS), Cramer–von Mises (CM) and Anderson–Darling (AD), verifying the hypothesis of uniformity.

In order to interpret the results of the tests we compare them with the corresponding critical values C_α (for the same significance level α). When the value of the test is less than the corresponding value C_α we accept the fit as adequate (there is no reason to reject the null hypothesis). The critical values C_α of the tests given a significance level α (e.g. $\alpha = 0.05$) can be easily found in the literature, see e.g. [2] and [11].

We conducted in [14] empirical studies for the losses occurred in Pumped Storage Power Plants (PSPP) Co. in Poland between 1991 and 1999. Distributions were fitted using the moments and maximum likelihood estimation. The results of the parameter estimation and test statistics are presented in Tab. 1. The Burr, lognormal and Weibull distributions passed all tests (the test statistics which are lower than the corresponding values C_α are in boldface). The expectation does not exist in the Burr case ($\alpha = 0.1416$ and $\tau = 2.7416$), thus we exclude it. Without a thorough examination it is impossible to say which distribution, lognormal or Weibull better fits the data as the results of the tests are quite similar. We will not conduct such analysis as it is beyond the scope of this article and we now choose both lognormal and Weibull distributions for the illustration of the obtained theoretical results.

The premium under no deductible corresponding to the lognormal distribution is nearly PLN 2.3 million and to Weibull PLN 1.7 million (we note that in order to obtain an annual premium we would have to multiply it by mean number of losses per year). The expectations do not tally because the maximum likelihood estimation of parameters of different distributions does not imply equality of moments.

Now, we are going to show the influence of incorporating different deductibles on the premium. Let us first concentrate on franchise and fixed amount deductibles. Figure 6 depicts the comparison of the two corresponding premiums and the effect of increasing the deductible a and b, cf. formulae (52) and (53). Evidently $P \geqslant P_{FD} \geqslant P_{FAD}$. Moreover, we can see that deducible of about PLN 10 milion in the lognormal case and PLN 5 milion in the Weibull case reduces P_{FAD} by half. Figure 6a (lognormal case) and Figure 6b (Weibull case) are similar, however we note that the differences do not lie in shifting or scaling. The same is true for the rest of considered deductibles.

The proportional deductible influences the premium in an obvious manner, that is pro rata (e.g. $c = 0.25$ results in cutting the premium by a quarter).

Figure 7 depicts the effect of parameters c, m_1 and m_2 of the limited proportional deductible. It is easy to see that $P_{LPD(c,m_1,m_2)}$ is a decreasing function of these parameters.

Finally, Figure 8 depicts the influence of parameters d_1 and d_2 of the disappearing deductible. Clearly, $P_{DD(d_1,d_2)}$ is a decreasing function of the parameters and we can observe that the effect of increasing d_2 is rather minor.

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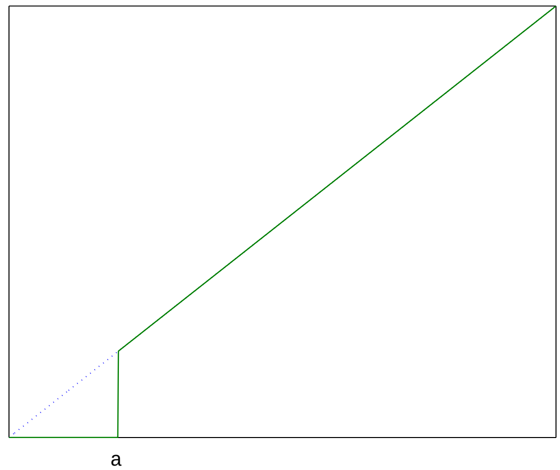


Figure 1. The payment function under the franchise (solid line) and no deductible (dotted line).

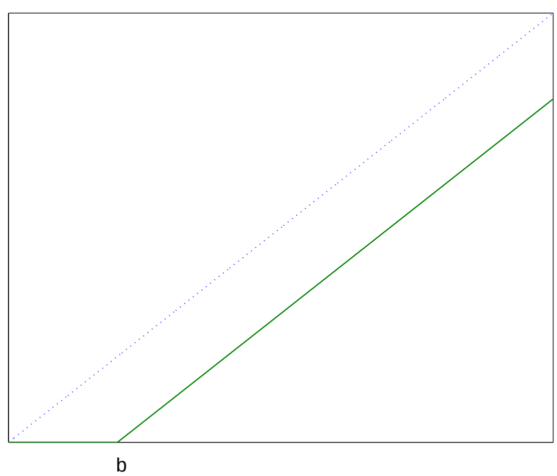


Figure 2. The payment function under the fix amount (solid line) and no deductible (dotted line).

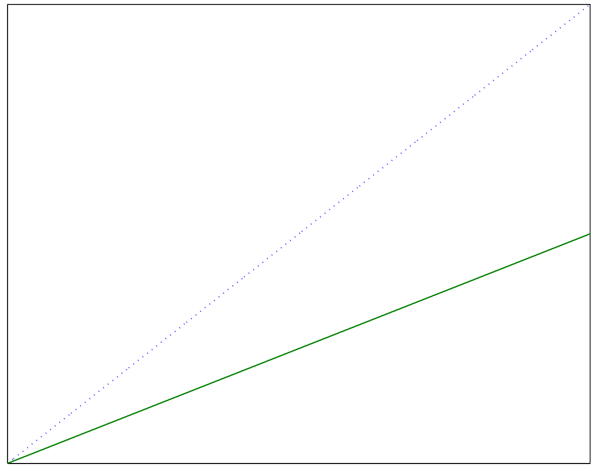


FIGURE 3. The payment function under the proportional (solid line) and no deductible (dotted line).

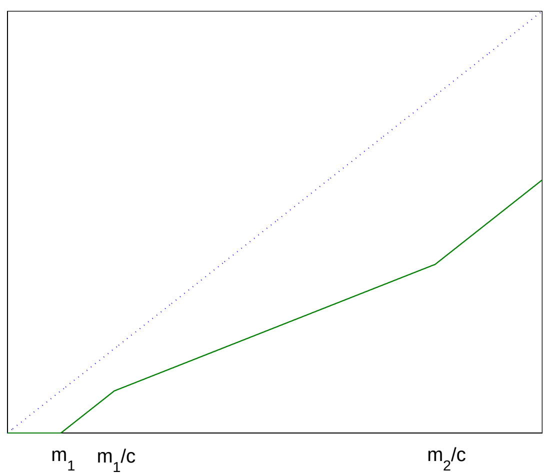


FIGURE 4. The payment function under the limited proportional (solid line) and no deductible (dotted line).

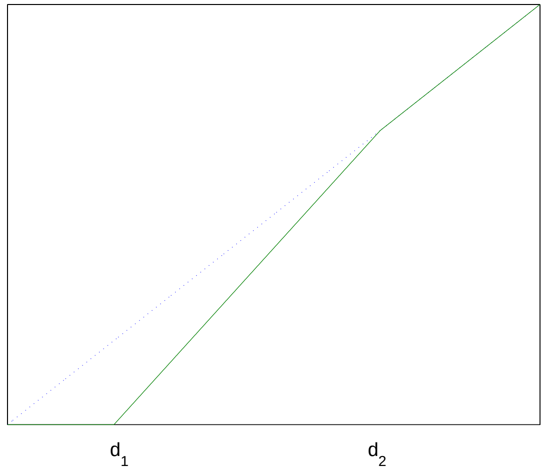
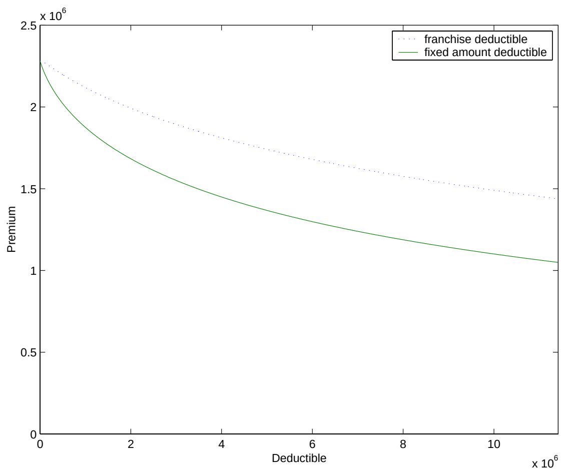
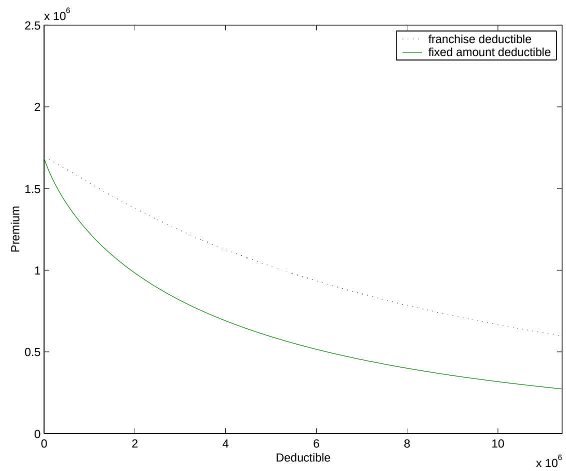


FIGURE 5. The payment function under the disappearing (solid line) and no deductible (dotted line).

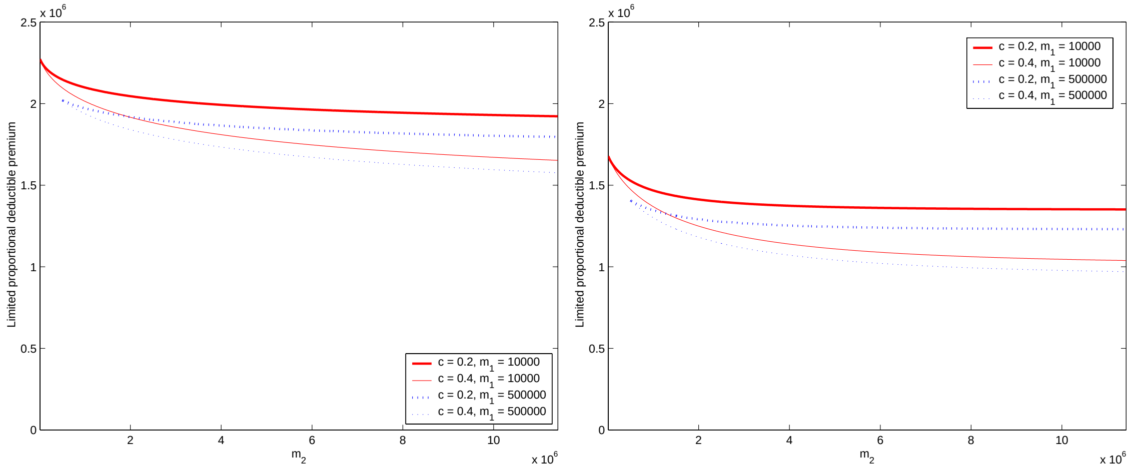


(a)



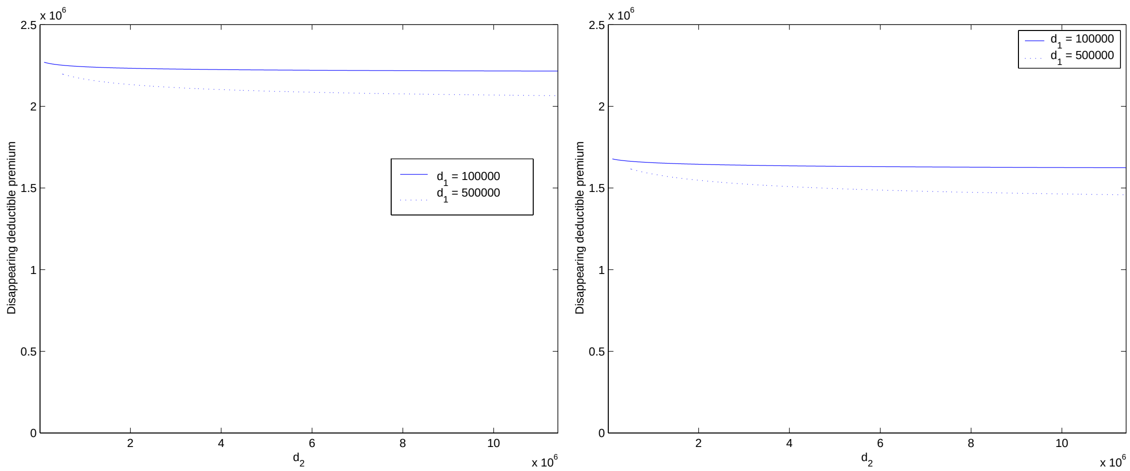
(b)

FIGURE 6. The premium under franchise and fixed amount deductibles. The lognormal (a) and Weibull (b) case based on the PSPP Co. loss data.



(a) (b)

FIGURE 7. The premium under the limited proportional deductible. The lognormal (a) and Weibull (b) case based on the PSPP Co. loss data.



(a) (b)

FIGURE 8. The premium under the disappearing deductible. The lognormal (a) and Weibull (b) case based on the PSPP Co. loss data.

Table 1. Parameter estimates and test statistics for the power company loss data. Parameter estimates were obtained through moment or maximum likelihood estimation

Distributions:	Lognormal	Pareto	Burr	Weibull	Gamma
Parameters:	μ=12.3172 σ=2.1558	α=2.5262 λ=2.9842e+006	α=0.1416 λ=4.7176e+011 τ=2.7416	α=0.4525 β=6.8915e+005	α=0.31536 β=6.2005e+006
Test values (in brackets: critical values for $\alpha = 0.05$):					
χ^2 (9.4877)	**4**	11	**2**	**2**	11
KS (0.4094)	**0.2286**	0.5125	**0.1610**	**0.2465**	**0.2879**
CM (0.4531)	**0.0623**	0.6222	**0.0372**	**0.0805**	**0.1416**
AD (2.492)	**0.3578**	4.82	**0.2218**	**0.4841**	**0.7809**

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